NORTH SYDNEY GIRLS HIGH SCHOOL

2017 TRIAL HSC EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black pen
- Board approved calculators may be used
- A reference sheet has been provided
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 2 – 6

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II Pages 7 – 15

90 Marks

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

NAME:______________________________ **TEACHER:**___________________

STUDENT NUMBER:__________________________

Question	1-10	11	12	13	14	15	16	Total
Mark	/10	/15	/15	/15	/15	/15	/15	/100

Section I

10 marks
Attempt Questions 1−10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet for Questions 1–10.

---

1 If k is a real number, then what is $\dfrac{k^2+4}{2-ki}$ equal to?

(A) $2-ki$

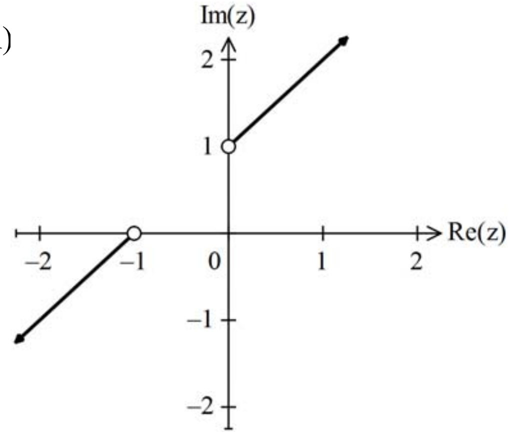
(B) $k-2i$

(C) $2+ki$

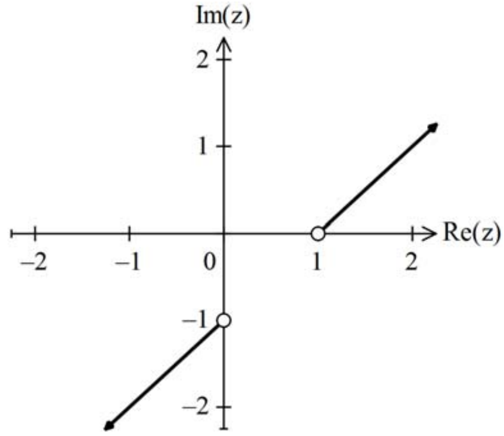
(D) $k+2i$

2 Which of the following Argand diagrams describes the locus defined by $\arg(z-i)=\arg(z+1)$?

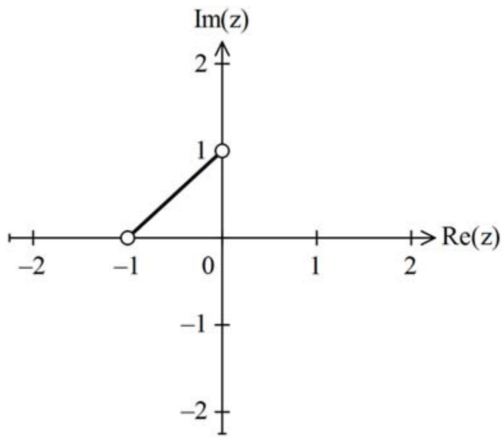
(A)



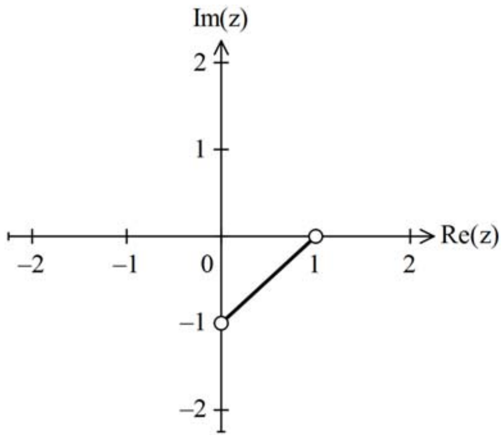
(B)



(C)



(D)



3 The roots of the polynomial $x^3 - 2x^2 + 5x + 4 = 0$ are α, β and γ.
What is the value of $\alpha^2 + \beta^2 + \gamma^2$?

(A) 4

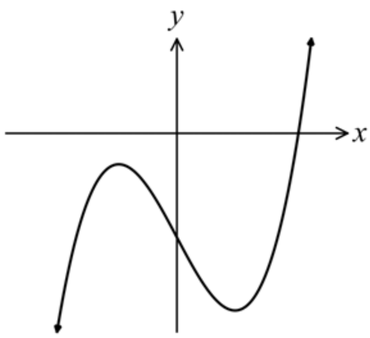
(B) 25

(C) –6

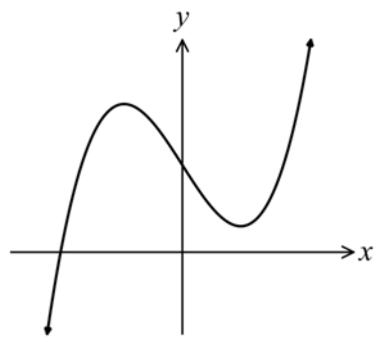
(D) 14

4 The polynomial $P(x) = x^3 - 11x - 20$ has a zero at $x = -2 + i$. Which of the graphs below could be the graph of $y = P(x)$?

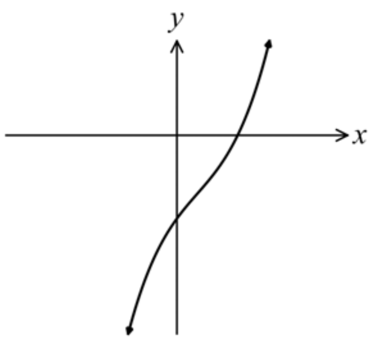
(A)



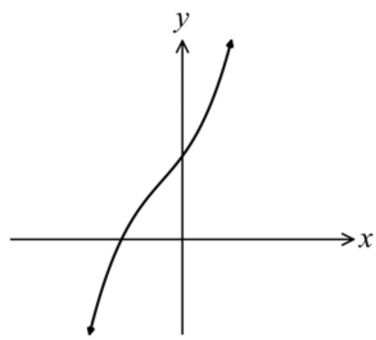
(B)



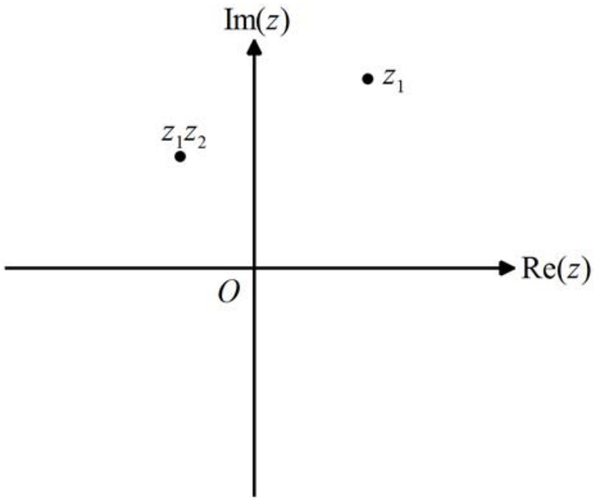
(C)



(D)



5 Let $z_1 = r_1 \text{cis}\alpha$ and $z_2 = r_2 \text{cis}\beta$ where r_1 and r_2 are real numbers. z_1 and $z_1 z_2$ are shown in the Argand diagram below.



Which of the following is necessarily true?

(A) $r_1 > 1$

(B) $r_2 > 1$

(C) $\left|\dfrac{z_1}{z_2}\right| > r_1$

(D) $r_2 < |z_1 z_2|$

6 P is an extremity of the minor axis of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with foci at S and S'. If SPS' is a right-angled triangle, what is the eccentricity of the ellipse?

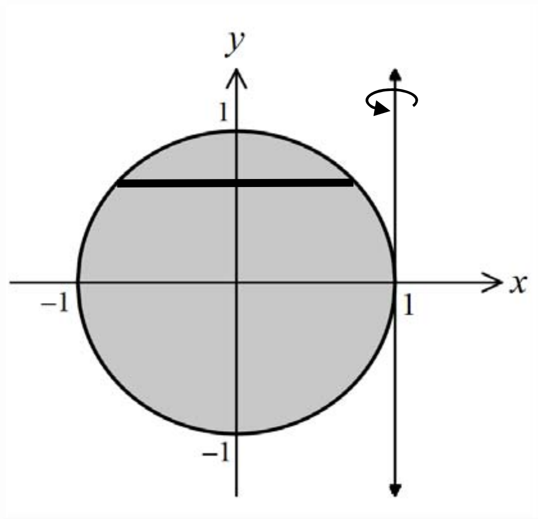
(A) $\dfrac{1}{2}$

(B) $\dfrac{1}{\sqrt{2}}$

(C) $\dfrac{\sqrt{3}}{2}$

(D) $\dfrac{2}{\sqrt{3}}$

7 The region bounded by the circle $x^2 + y^2 = 1$ is rotated about the line $x = 1$.



What is the volume of the solid of revolution formed?

(A) $V = 4\pi \int_0^1 (1 - y^2)^{\frac{1}{2}}\, dy$

(B) $V = 8\pi \int_0^1 (1 - y^2)^{\frac{1}{2}}\, dy$

(C) $V = 4\pi \int_0^1 (1 - y^2)\, dy$

(D) $V = 8\pi \int_0^1 (1 - y^2)\, dy$

8 What is the range of the function $f(x) = (x^2 - 1)\sin^{-1}(x - 1)$?

(A) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(B) $\frac{\pi}{2} \le y \le \frac{3\pi}{2}$

(C) $0 \le y \le \frac{\pi}{2}$

(D) $0 \le y \le \frac{3\pi}{2}$

9 If $a > b$ and $k < 0$, which of the following must be true?

I. $a^2 > b^2$

II. $a + k > b + k$

III. $\dfrac{a}{k^2} > \dfrac{b}{k^2}$

(A) I and II only

(B) II and III only

(C) I and III only

(D) I, II and III

10 Which expression must be equal to $\displaystyle\int_0^a \left[f(a-x) + f(a+x) \right] dx$?

(A) $\displaystyle\int_0^a f(x)\,dx$

(B) $\displaystyle\int_0^{2a} f(x)\,dx$

(C) $\displaystyle 2\int_0^a f(x)\,dx$

(D) $\displaystyle\int_{-a}^a f(x)\,dx$

Section II

Total marks − 90
Attempt Questions 11−16
Allow about 2 hour 45 minutes for this section.
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 to 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the complex number $z = \sqrt{3} - 3i$.

(i) Express z in modulus-argument form. **2**

(ii) Write down the argument of z^4. **1**

(iii) Hence, write z^4 in the form $x + iy$, where x and y are real. **1**

(b) Find $\int \sin^3 x \, dx$. **2**

(c) Sketch the conic defined by $|z-1| + |z+1| = 4$ showing intercepts, foci and directrices. **2**

(d) Consider the polynomial equation $P(z) = z^4 - 4z^3 + 7z^2 - 4z + 6$.

(i) Show that $z = i$ is a zero of this polynomial. **1**

(ii) Hence, write down a quadratic real factor of $P(z)$. **1**

(iii) Find all the roots of $P(z) = 0$. **2**

(e) A relation is defined by the equation $\sin x + \cos y = \frac{1}{2}$,

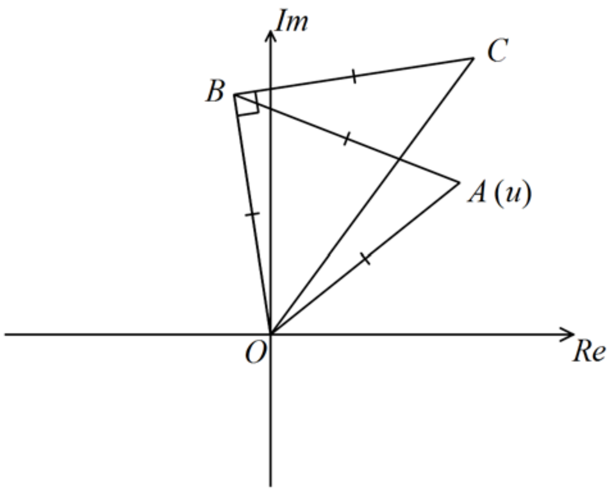
where $-\pi < x < \pi$ and $-\pi < y < \pi$.

(i) Show that $\frac{dy}{dx} = \frac{\cos x}{\sin y}$. **1**

(ii) Find the coordinates of the points where $\frac{dy}{dx} = 0$. **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a)



In the Argand diagram above, $OA = OB = AB = BC$ and $OB \perp BC$.
A represents the complex number u.

(i) Find the complex number represented by B. **1**

(ii) Hence, or otherwise, find an expression in terms of u for the complex number represented by C. **2**

(b) (i) Find real numbers a, b and c such that **2**

$$\frac{3-x}{(x^2+1)(1-2x)} = \frac{ax+b}{x^2+1} + \frac{c}{1-2x}.$$

(ii) Hence find $\displaystyle\int \frac{3-x}{(x^2+1)(1-2x)}\,dx$. **2**

(c) (i) Sketch the curve $f(x) = \dfrac{x-3}{x^2+x-2}$ showing all intercepts and asymptotes. **3**

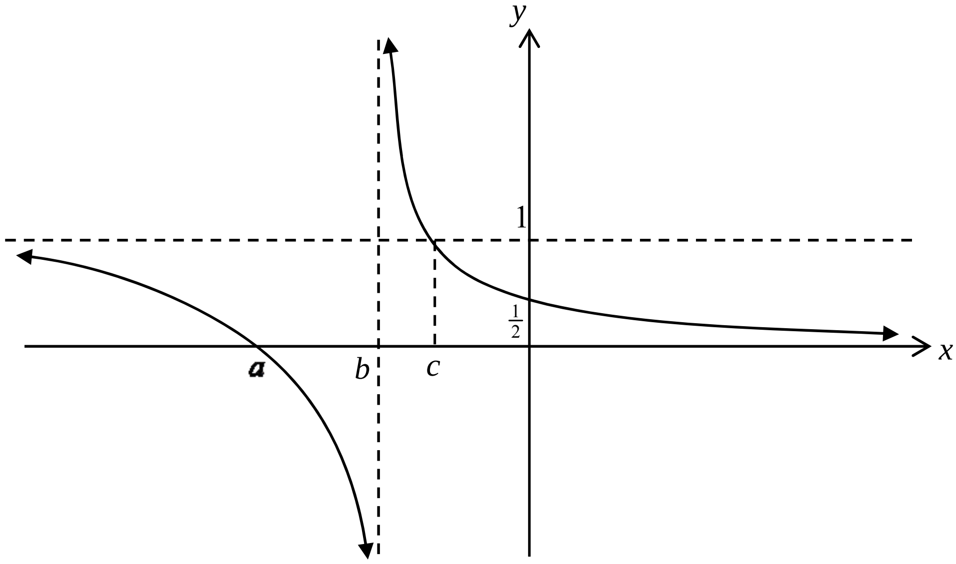
You are not required to find stationary points.

(ii) Hence solve $\dfrac{|x|-3}{x^2+|x|-2} \geq 0$. **1**

Question 12 continues on Page 9

Question 12 (continued)

(d) Given below is the graph of $y = f(x)$. The line $y = 1$ is a horizontal asymptote and $x = b$ is a vertical asymptote. The x-intercept is at $x = a$ and the y-intercept is at $y = \frac{1}{2}$.



Neatly sketch the graphs of the following showing all important information, including the location of a, b and c:

(i) $\dfrac{1}{f(x)}$ **2**

(ii) $\tan^{-1}\left(f(x)\right)$ **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) If $u = 6 + ki$ and $v = 4 + ki$, find k if $\arg(uv) = \dfrac{\pi}{4}$. **3**

(b) $P(a\sec\theta, b\tan\theta)$ is a point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

(i) Show that the tangent at P has the equation **2**

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$$

(ii) The tangent at P meets the asymptotes of the hyperbola at Q and R. **3**
Show that $PQ = PR$.

(c) If α, β, γ are the roots of the equation $x^3 + 2x^2 + 3x - 4 = 0$, find a polynomial equation whose roots are:

(i) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$ and $\dfrac{1}{\gamma}$ **2**

(ii) α^2, β^2 and γ^2 **2**

(d) A sequence is defined by **3**
$a_1 = 5,\ a_2 = 13$ and $a_{n+2} = 5a_{n+1} - 6a_n$ for all natural numbers n.

Use Mathematical Induction to prove that $a_n = 2^n + 3^n$.

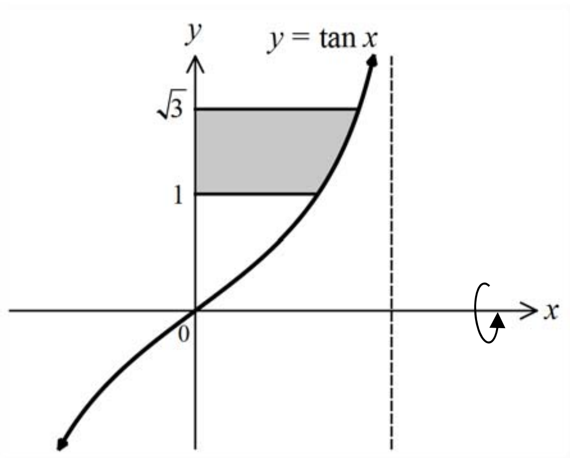
Question 14 (15 marks) Use a SEPARATE writing booklet

(a) Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate **3**

$$\int_0^{\frac{\pi}{3}} \frac{1}{1+\cos x - \sin x}\,dx \quad .$$

(b) (i) Use integration by parts to find $\int x \tan^{-1} x \, dx$. **2**

(ii) The region bounded by the curve $y = \tan x$, the lines $y = 1$, $y = \sqrt{3}$ and the y-axis is rotated about the x-axis. **2**



Use the method of cylindrical shells to find the volume of the solid of revolution formed.

(c) (i) Show that $a^2 + 9b^2 \geq 6ab$, where a and b are real numbers. **1**

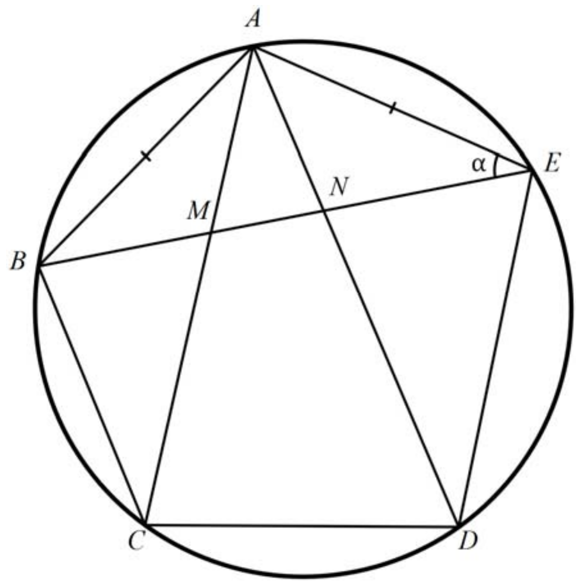
(ii) Hence, or otherwise, show that $a^2 + 5b^2 + 9c^2 \geq 3(ab + bc + ac)$. **2**

(iii) Hence if $a > b > c > 0$, show that $a^2 + 5b^2 + 9c^2 > 9bc$. **1**

Question 14 continues on Page 12

Question 14 (continued)

(d) $ABCDE$ is a pentagon inscribed in a circle and $AB = AE$.
BE meets AC and AD at M and N respectively. Let $\angle BEA = \alpha$.



(i) Explain why $\angle ACE = \alpha$. **1**

(ii) Prove that $CDNM$ is a cyclic quadrilateral. **3**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) Given that $p(x) = x^3 - 3ax^2 + b$ has a double zero, where a and b are non-zero real numbers, show that $4a^3 - b = 0$. **2**

(b) $P\left(cp, \frac{c}{p}\right)$ lies on the hyperbola $xy = c^2$.

(i) Show that the equation of the normal to the hyperbola at P is given by $px - \frac{y}{p} = c\left(p^2 - \frac{1}{p^2}\right)$. **2**

(ii) The equation of the tangent at P is $x + p^2 y = 2cp$. DO NOT PROVE THIS. **3**

The normal at P cuts the x-axis at Q and the tangent at P cuts the y-axis at R. M is the midpoint of QR.

Find the equation of the locus of M as P moves on the hyperbola.

(c) Let $I = \displaystyle\int_{\frac{1}{a}}^{a} \frac{f(x)}{x\left(f(x) + f\left(\frac{1}{x}\right)\right)} dx.$

(i) Use a suitable substitution to show that $I = \displaystyle\int_{\frac{1}{a}}^{a} \frac{f\left(\frac{1}{x}\right)}{x\left(f(x) + f\left(\frac{1}{x}\right)\right)} dx$. **2**

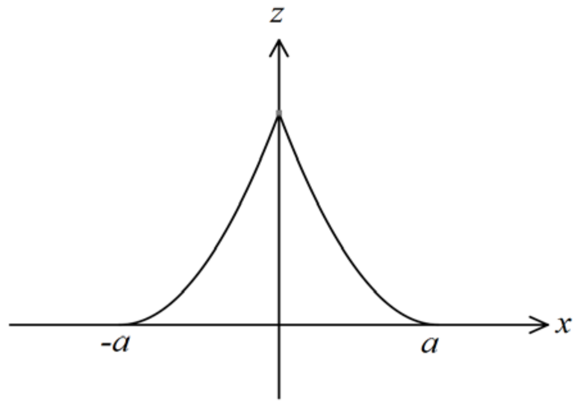
(ii) Hence, or otherwise, evaluate $\displaystyle\int_{\frac{1}{2}}^{2} \frac{\sin x}{x\left(\sin x + \sin\frac{1}{x}\right)} dx$. **2**

Question 15 continues on Page 14

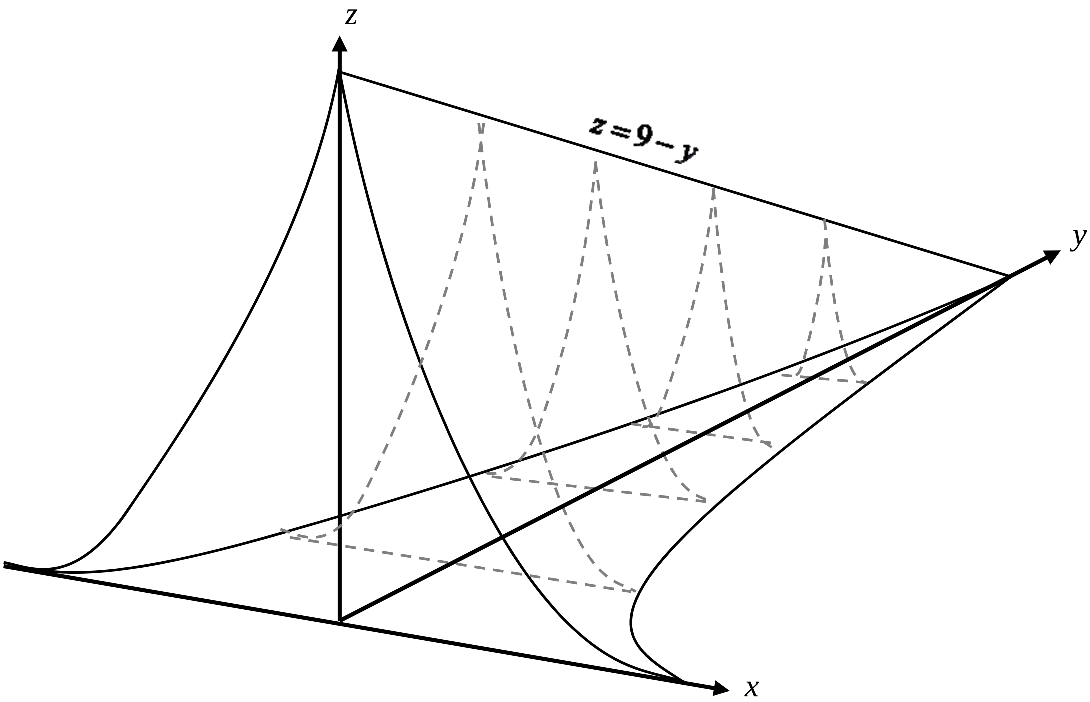
Question 15 (continued)

(d) (i) Find the area bounded by the curve $y=(x-a)^2$ and the coordinate axes, where $a>0$. **1**

(ii) Cross-sections of a solid perpendicular to the base are sections of two parabolas $z=(x-a)^2$ and $z=(x+a)^2$ as shown below. **3**



The heights of the cross-sections are bounded by the line $z=9-y$.



Find the volume of the solid.

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) Consider $I_n = \int \left(x^2+1\right)^{-n} dx, \quad n > 0$.

(i) Show that $I_{n+1} = \dfrac{x\left(x^2+1\right)^{-n}}{2n} + \dfrac{2n-1}{2n} I_n$. **3**

(ii) Find I_2. **1**

(b) (i) Using the result $\cos(A+B) - \cos(A-B) = -2\sin A \sin B$, or otherwise, **2**

show that $\sin\theta \sum_{r=1}^{n} \sin 2r\theta = \sin(n+1)\theta \sin n\theta$.

(ii) Hence show that $\sum_{r=1}^{8} \sin\dfrac{r\pi}{9} = \cot\dfrac{\pi}{18}$. **2**

(c) (i) Show that $\log_e x \le x - 1$ for $x > 0$. **2**

(ii) Show that $\log_e\left(\dfrac{c_1 c_2 \ldots c_n}{\mu^n}\right) \le \dfrac{c_1 + c_2 + \ldots + c_n}{\mu} - n$, where $c_1, c_2, \ldots, c_n > 0$ and $\mu > 0$. **2**

(iii) Hence if $\mu = \dfrac{c_1 + c_2 + \ldots. + c_n}{n}$, show that $\sqrt[n]{c_1 c_2 \ldots c_n} \le \dfrac{c_1 + c_2 + \ldots + c_n}{n}$. **2**

(iv) Hence use part (iii) to find a lower bound for $\dfrac{101}{103} + \dfrac{103}{105} + \dfrac{105}{107} + \ldots. + \dfrac{197}{199} + \dfrac{199}{101}$. **1**

End of paper

1 If k is a real number, then what is $\dfrac{k^2+4}{2-ki}$ equal to?

(A) $2-ki$

(B) $k-2i$

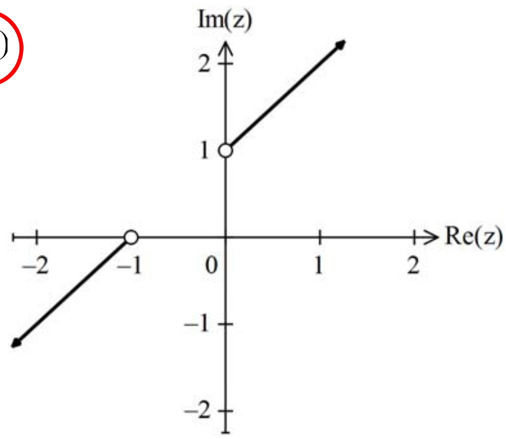
(C) $2+ki$

(D) $k+2i$

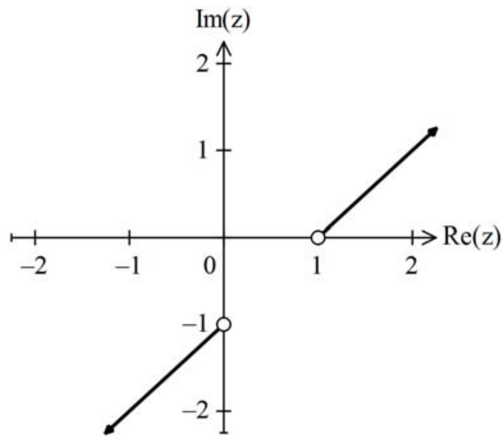
$$\frac{k^2+4}{2-ki}\times\frac{2+ki}{2+ki}=\frac{\cancel{(k^2+4)}\times(2+ki)}{\cancel{4+k^2}}$$
$$=2+ki$$

2 Which of the following Argand diagrams describes the locus defined by $\arg(z-i)=\arg(z+1)$?

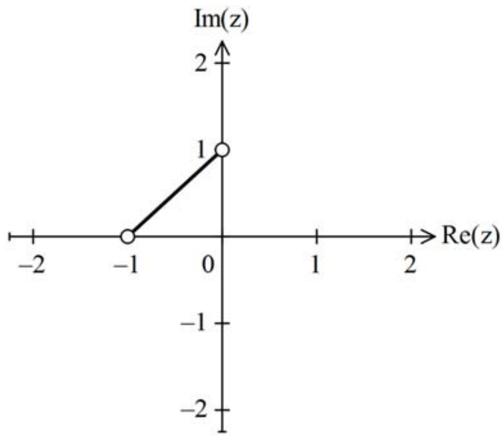
(A)



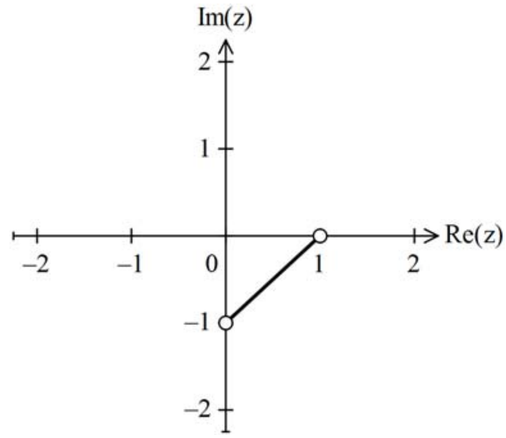
(B)



(C)



(D)



3 The roots of the polynomial $x^3 - 2x^2 + 5x + 4 = 0$ are α, β and γ.

What is the value of $\alpha^2 + \beta^2 + \gamma^2$?

(A) 4

(B) 25

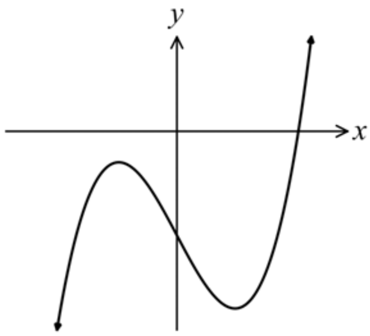
(C) –6

(D) 14

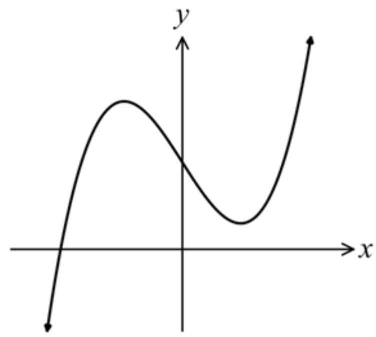
$$\begin{aligned}\alpha^2+\beta^2+\gamma^2 &= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)\\ &= (-2)^2 - 2(5) = 4-10\\ &= -6\end{aligned}$$

4 The polynomial $P(x) = x^3 - 11x - 20$ has a zero at $x = -2 + i$. Which of the graphs below could be the graph of $y = P(x)$?

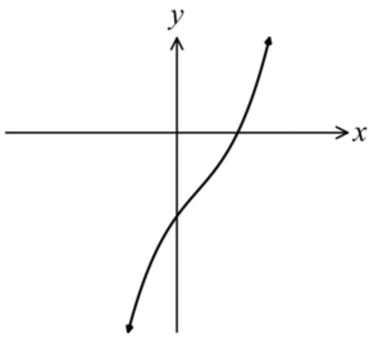
(A)



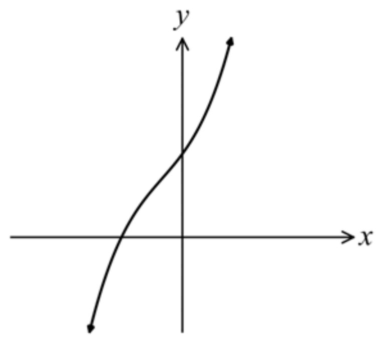
(B)



(C)



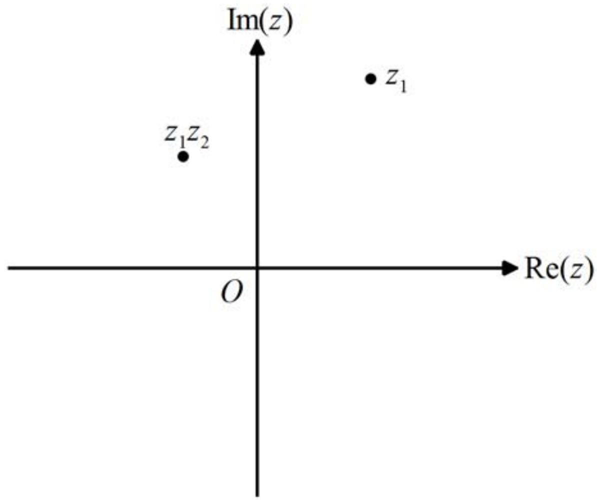
(D)



The graph has a negative y-intercept so must be A or C. Solving for stationary points, $P'(x) = 0 \Rightarrow 3x^2 - 11 = 0 \Rightarrow x = \pm\sqrt{\dfrac{11}{3}}$. So it has stationary points. Hence, A.

5 Let $z_1 = r_1 \text{cis}\alpha$ and $z_2 = r_2 \text{cis}\beta$ where r_1 and r_2 are real numbers.

z_1 and $z_1 z_2$ are shown in the Argand diagram below.



Which of the following is necessarily true?

(A) $r_1 > 1$

(B) $r_2 > 1$

(C) $\left|\dfrac{z_1}{z_2}\right| > r_1$

(D) $r_2 < |z_1 z_2|$

There is no scale, so neither r_1 or r_2 needs to be greater than 1. D looks like it has to be true, but if both r_1 and r_2 are smaller than 1, then $|z_1 z_2| = r_1 r_2$ will be smaller than r_2. Rearranging C we get $|z_1| > r_1 |z_2|$ or $|z_1| > |z_1 z_2|$ which is true from the diagram. Hence, C.

6 P is an extremity of the minor axis of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with foci at S and S'. If SPS' is a right-angled triangle, what is the eccentricity of the ellipse?

(A) $\dfrac{1}{2}$

(B) $\dfrac{1}{\sqrt{2}}$

(C) $\dfrac{\sqrt{3}}{2}$

(D) $\dfrac{2}{\sqrt{3}}$

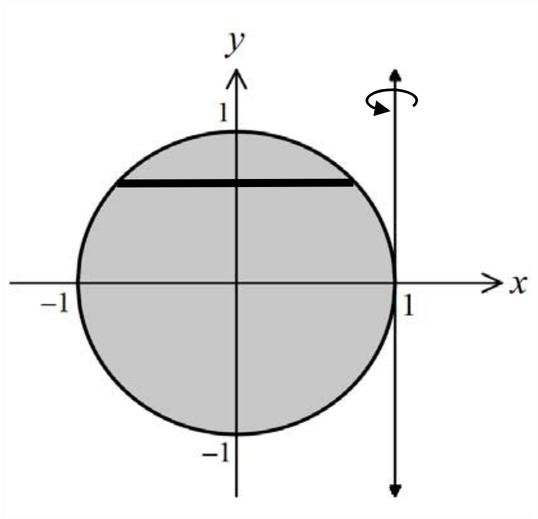
$P = (0, \pm b)$ and $m_{PS} \times m_{PS'} = -1$

$$\frac{b-0}{0-ae} \times \frac{b-0}{0+ae} = -1 \Rightarrow \frac{b^2}{-a^2e^2} = -1$$

$$b^2 = a^2e^2 \Rightarrow e^2 = \frac{b^2}{a^2}$$

$$e^2 = 1 - e^2 \text{ or } 2e^2 = 1 \Rightarrow e = \frac{1}{\sqrt{2}}$$

7 The region bounded by the circle $x^2+y^2=1$ is rotated about the line $x=1$.



What is the volume of the solid of revolution formed?

(A) $V=4\pi\int_0^1 (1-y^2)^{\frac{1}{2}}\,dy$

(B) $V=8\pi\int_0^1 (1-y^2)^{\frac{1}{2}}\,dy$

(C) $V=4\pi\int_0^1 (1-y^2)\,dy$

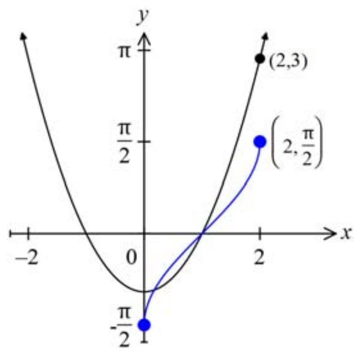
(D) $V=8\pi\int_0^1 (1-y^2)\,dy$

$x^2+y^2=1 \Rightarrow x=\pm\sqrt{1-y^2}$

$R=1+\sqrt{1-y^2}$ and $r=1-\sqrt{1-y^2}$

$$V=\pi\int_{-1}^{1}(R^2-r^2)\,dy$$
$$=2\pi\int_0^1 (R+r)(R-r)\,dy$$
$$=2\pi\int_0^1 (2)\left(2\sqrt{1-y^2}\right)dy$$
$$=8\pi\int_0^1 (1-y^2)^{\frac{1}{2}}\,dy$$

8 What is the range of the function $f(x)=(x^2-1)\sin^{-1}(x-1)$?



(A) $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$

(B) $\frac{\pi}{2}\le y\le\frac{3\pi}{2}$

(C) $0\le y\le\frac{\pi}{2}$

(D) $0\le y\le\frac{3\pi}{2}$

From the graphs of $y=x^2-1$ and $y=\sin^{-1}(x-1)$ both are negative or both are positive. So the range is always positive. When both are 0, $y=0$, so not B.

Finally, looking at endpoints of the domain of $y=\sin^{-1}(x-1)$, the answer is D.

9 If $a > b$ and $k < 0$, which of the following must be true?

I. $a^2 > b^2$

II. $a + k > b + k$

III. $\frac{a}{k^2} > \frac{b}{k^2}$

(A) I and II only

(B) II and III only

(C) I and III only

(D) I, II and III

I is true only if a and b are positive.

II is true regardless of whether a,b, or k are positive.

III is true as $k^2 > 0$ when $k < 0$

Therefore II and III are true, so B.

10 Which expression must be equal to $\int_0^a \left[f(a-x) + f(a+x) \right] dx$?

(A) $\int_0^a f(x)\,dx$

(B) $\int_0^{2a} f(x)\,dx$

(C) $2\int_0^a f(x)\,dx$

(D) $\int_{-a}^a f(x)\,dx$

$\int_0^a f(a-x)\,dx = \int_0^a f(x)\,dx$ by symmetry in $x = \frac{a}{2}$.

$y = f(a+x)$ is $y = f(x)$ shifted right by a units.

$\int_0^a f(a+x)\,dx = \int_a^{2a} f(x)\,du$.

Hence,

$$\begin{aligned}\int_0^a \left[f(a-x) + f(a+x) \right] dx &= \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx \\ &= \int_0^{2a} f(x)\,dx\end{aligned}$$

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the complex number $z = \sqrt{3} - 3i$.

(i) Express z in modulus-argument form. **2**

(ii) Write down the argument of z^4. **1**

(iii) Hence write z^4 in the form $x + iy$, where x and y are real. **1**

(i) $|z| = \sqrt{\sqrt{3}^2 + (-3)^2} = \sqrt{12}$

$\tan(\arg z) = \dfrac{-3}{\sqrt{3}} = -\sqrt{3} \Rightarrow \arg z = -\dfrac{\pi}{3}$ (4th quadrant)

$z = 2\sqrt{3}\left(\cos\left(-\dfrac{\pi}{3}\right) + i\sin\left(-\dfrac{\pi}{3}\right)\right)$

(ii) $\arg\left(z^4\right) = 4\arg(z) = 4\left(-\dfrac{\pi}{3}\right) = -\dfrac{4\pi}{3}$ or $\dfrac{2\pi}{3}$

(iii) $z^4 = \left(2\sqrt{3}\right)^4\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$

$= 144\left(-\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right)$

$= -72 + 72\sqrt{3}i$

(b) Find $\int \sin^3 x\, dx$. **2**

$$\int \sin^3 x dx = \int \left(1 - \cos^2 x\right)\sin x dx$$

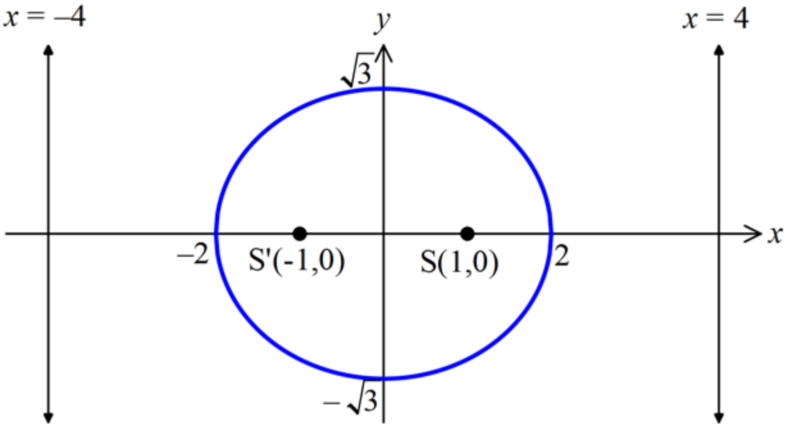
$$= \int \sin x dx + \int \cos^2 x(-\sin x)\, dx$$

$$= -\cos x + \frac{\cos^3 x}{3} + C$$

(c)	Sketch the conic defined by $\lvert z-1\rvert+\lvert z+1\rvert=4$ showing intercepts, foci and directrices.	**2**

Using $PS+PS'=2a$, this is an ellipse with $2a=4$ and $S(1,0)$ and $S'(-1,0)$.

Then $a=2$; and $ae=1 \Rightarrow e=\frac{1}{2}$. Then $\frac{a}{e}=4$ and $b^2=a^2\left(1-e^2\right)=4\left(1-\frac{1}{4}\right)=3$



(d)	Consider the polynomial equation $P(z)=z^4-4z^3+7z^2-4z+6$.	
	(i) Show that $z=i$ is a zero of this polynomial.	**1**
	(ii) Hence write down a quadratic real factor of $P(z)$.	**1**
	(iii) Find all the roots of $P(z)=0$.	**2**

(i) $P(i)=i^4-4i^3+7i^2-4i+6=1+4i-7-4i+6=0$. Therefore, $z=i$ is a zero or $z=i$ is a root of $P(z)=0$.

(ii) As the polynomial equation $P(z)=0$ has real coefficients, complex roots occur in conjugate pairs. Thus, $z=-i$ is also a root and $(z-i)(z+i)=z^2-i^2=z^2+1$ is a factor of $P(z)$.

(iii) Hence, $P(z)=z^4-4z^3+7z^2-4z+6=\left(z^2+1\right)\left(z^2-4z+6\right)$ by inspection

$z^2-4z+6=0$

$(z-2)^2=-2$ by completing the square

$z-2=\pm i\sqrt{2}$

$z=2\pm i\sqrt{2}$

Therefore the roots are $\pm i,\ 2\pm i\sqrt{2}$.

(e) A relation is defined by the equation $\sin x + \cos y = \frac{1}{2}$,

where $-\pi < x < \pi$ and $-\pi < y < \pi$.

(i) Show that $\frac{dy}{dx} = \frac{\cos x}{\sin y}$. **1**

(ii) Find the coordinates of the points where $\frac{dy}{dx} = 0$. **2**

(i) Differentiating implicitly wrt x

$$\cos x - \sin y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\cos x}{-\sin y} = \frac{\cos x}{\sin y}$$

(ii) $\frac{dy}{dx} = 0 \Rightarrow \cos x = 0$ and $\sin y \neq 0$.

$\cos x = 0 \Rightarrow x = \pm\frac{\pi}{2} \quad (-\pi < x < \pi)$.

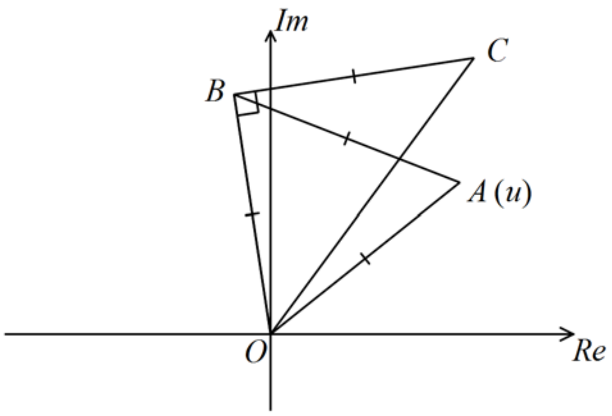
$x = \frac{\pi}{2} \Rightarrow \sin\frac{\pi}{2} + \cos y = \frac{1}{2}$ $\qquad$ $x = -\frac{\pi}{2} \Rightarrow \sin\left(-\frac{\pi}{2}\right) + \cos y = \frac{1}{2}$

$\cos y = -\frac{1}{2} \Rightarrow y = \pm\frac{2\pi}{3}$ $\qquad$ $\cos y = \frac{3}{2} \Rightarrow$ no solutions

Therefore, $\frac{dy}{dx} = 0$ at $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and $\left(\frac{\pi}{2}, -\frac{2\pi}{3}\right)$.

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a)



In the Argand diagram above, $OA = OB = AB = BC$ and $OB \perp BC$.
A represents the complex number u.

(i) Find the complex number represented by B. **1**

(ii) Hence, or otherwise, find the complex number represented by C. **2**

(i) $\angle AOB = \frac{\pi}{3}$ (equilateral triangle)

$\therefore B \equiv u.\text{cis}\left(\frac{\pi}{3}\right)$. As $\overrightarrow{OB}$ is obtained by rotating $\overrightarrow{OA}$ anticlockwise by $\frac{\pi}{3}$.

(ii) $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$

Now, $\overrightarrow{BC}$ is obtained by rotating $\overrightarrow{BO}$ anticlockwise by $\frac{\pi}{2}$.

$$\therefore C \equiv u\text{cis}\left(\frac{\pi}{3}\right) - i.u\text{cis}\left(\frac{\pi}{3}\right)$$

$$C \equiv (1-i)u\text{cis}\left(\frac{\pi}{3}\right)$$

Other valid approaches possible. For instance,

$\left|\overrightarrow{OC}\right|^2 = \left|\overrightarrow{OB}\right|^2 + \left|\overrightarrow{BC}\right|^2$ Pythag. So, $\left|\overrightarrow{OC}\right| = |u|\sqrt{2}$

$\angle AOC = \angle AOB - \angle BOC = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

$\overrightarrow{OC}$ is obtained by rotating $\overrightarrow{OA}$ anticlockwise by $\frac{\pi}{12}$. So, $\overrightarrow{OC} = u\sqrt{2}\text{cis}\left(\frac{\pi}{12}\right)$.

(b) (i) Find real numbers a, b and c such that **2**

$$\frac{3-x}{\left(x^2+1\right)\left(1-2x\right)}=\frac{ax+b}{x^2+1}+\frac{c}{1-2x}.$$

Comparing numerators on both sides, $3-x\equiv(ax+b)(1-2x)+c\left(x^2+1\right)$

$x=\frac{1}{2}$: $\frac{5}{2}=\frac{5c}{4}\Rightarrow c=2$

$x=0$: $3=b+c\Rightarrow b=1$

x^2coeff: $0=-2a+c\Rightarrow a=1$

$\therefore a=1,\quad b=1,\quad c=2$

(ii) Hence find $\int\frac{3-x}{\left(x^2+1\right)\left(1-2x\right)}dx$. **2**

$$\int\frac{3-x}{\left(x^2+1\right)\left(1-2x\right)}dx=\int\frac{x+1}{x^2+1}dx+\int\frac{2}{1-2x}dx \qquad \text{using (i)}$$

$$=\int\frac{x}{x^2+1}dx+\int\frac{1}{x^2+1}dx-\int\frac{-2}{1-2x}dx$$

$$=\frac{1}{2}\log_e\left(x^2+1\right)+\tan^{-1}x-\log_e\left(1-2x\right)+C$$

(c) (i) Sketch the curve $f(x)=\frac{x-3}{x^2+x-2}$ showing all intercepts and asymptotes. **3**

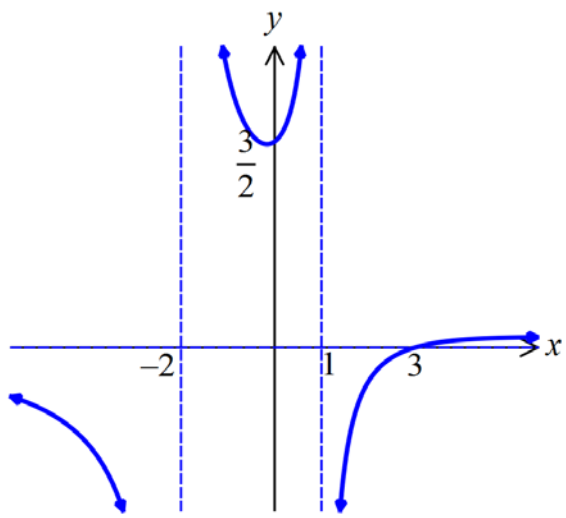
You are not required to find stationary points.

$f(x)=\frac{x-3}{x^2+x-2}=\frac{x-3}{(x+2)(x-1)}$

Vertical asymptotes at $x=-2, x=1$.

$\lim\limits_{x\to\pm\infty}f(x)=0$.

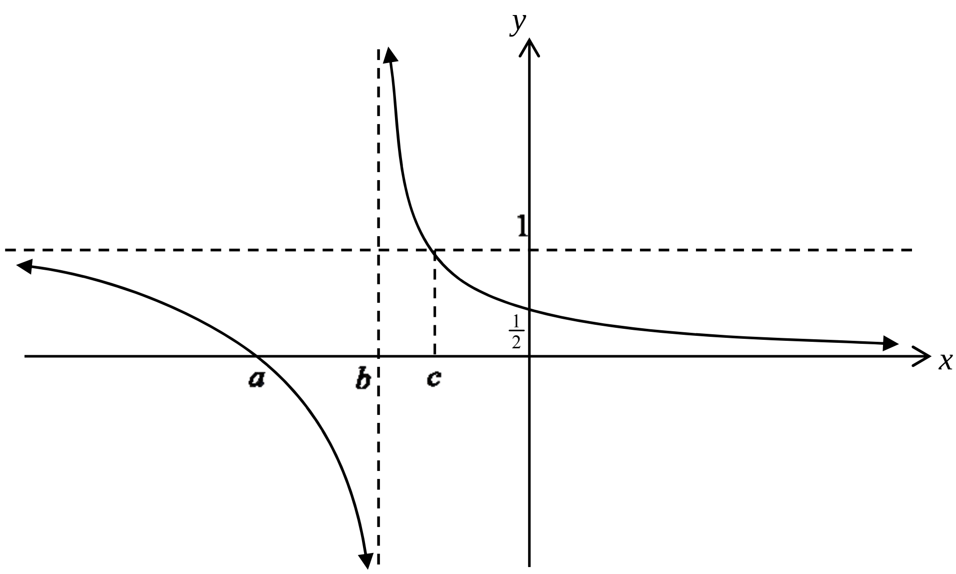
x-intercept is 3, y-intercept is $\frac{3}{2}$.



(ii) Hence solve $\frac{|x|-3}{x^2+|x|-2}\geq 0$. **1**

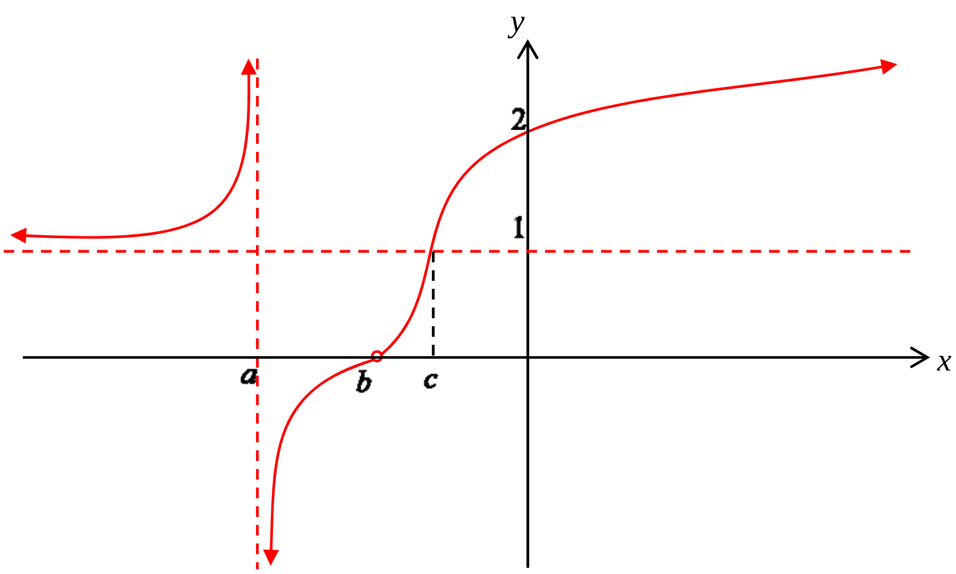
From the graph, using the symmetry transformation $y=f(|x|)$, solutions are $x\geq 3, x\leq -3, -1<x<1$

(d) Given below is the graph of $y = f(x)$. The line $y = 1$ is a horizontal asymptote and $x = b$ is a vertical asymptote. The x-intercept is at $x = a$ and the y-intercept is at $y = \frac{1}{2}$.

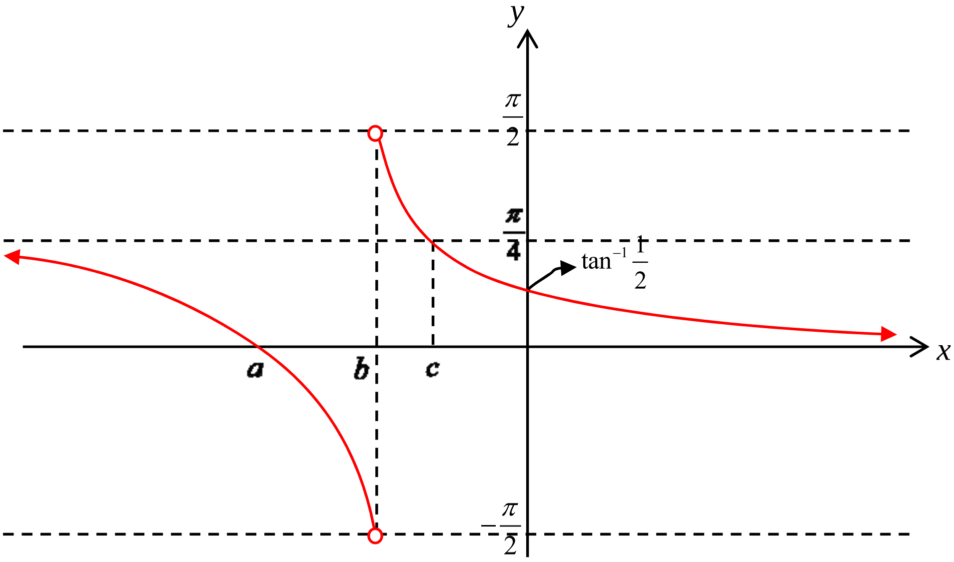


Neatly sketch the graphs of the following showing all important information, including the location of a, b and c:

(i) $\dfrac{1}{f(x)}$ **2**



(ii) $\tan^{-1}(f(x))$	**2**



Question 13 (15 marks) Use a SEPARATE writing booklet.

(a)	If $u=6+ki$ and $v=4+ki$, find k if $\arg(uv)=\dfrac{\pi}{4}$.	**3**

$$uv=(6+ki)(4+ki)=\left(24-k^2\right)+i(10k)$$

$$\arg(uv)=\frac{\pi}{4}$$

$$\tan(\arg(uv))=\frac{10k}{24-k^2}=1$$

$$10k=24-k^2$$

$$k^2+10k-24=0$$

$$(k+12)(k-2)=0$$

$$k=-12,2$$

But $k\neq -12$ as this gives $uv=-120-120k$ which is not in the first quadrant.
So $k=2$ is the only solution.

(b)	$P(a\sec\theta, b\tan\theta)$ is a point on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.	
	(i) Show that the tangent at P has the equation $\dfrac{x\sec\theta}{a}-\dfrac{y\tan\theta}{b}=1$	**2**

$$x=a\sec\theta\Rightarrow\frac{dx}{d\theta}=a\sec\theta\tan\theta$$

$$y=b\tan\theta\Rightarrow\frac{dy}{d\theta}=b\sec^2\theta$$

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{b\sec^2\theta}{a\sec\theta\tan\theta}=\frac{b\sec\theta}{a\tan\theta}$$

Equation of tangent at P is:

$$y-b\tan\theta=\frac{b\sec\theta}{a\tan\theta}(x-a\sec\theta)$$

$$ay\tan\theta-ab\tan^2\theta=bx\sec\theta-ab\sec^2\theta$$

$$bx\sec\theta-ay\tan\theta=ab\left(\sec^2\theta-\tan^2\theta\right)\qquad\div ab$$

$$\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1\qquad\text{as }\sec^2\theta-\tan^2\theta=1$$

(ii)	The tangent at P meets the asymptotes of the hyperbola at Q and R. Show that $PQ = PR$.	**3**

The asymptotes are $y = \pm\frac{b}{a}x$. Solving simultaneously with the tangent, we get

$$\frac{x\sec\theta}{a} - \frac{\left(\pm\frac{b}{a}x\right)\tan\theta}{b} = 1$$

$$\frac{x\sec\theta}{a} - \frac{\pm x\tan\theta}{a} = 1$$

$$\frac{x(\sec\theta \mp \tan\theta)}{a} = 1 \text{ or } x = \frac{a}{(\sec\theta \mp \tan\theta)}$$

So, $Q = \left(\frac{a}{\sec\theta - \tan\theta}, \frac{b}{\sec\theta - \tan\theta}\right)$ using $y = \frac{b}{a}x$ and

$R = \left(\frac{a}{\sec\theta + \tan\theta}, \frac{-b}{\sec\theta + \tan\theta}\right)$ using $y = -\frac{b}{a}x$

Now, if $PQ=PR$, then P is the midpoint of QR. Let M be the midpoint of QR.

$$x_M = \frac{1}{2}\left(\frac{a\sec\theta + \cancel{a\tan\theta} + a\sec\theta \cancel{- a\tan\theta}}{\sec^2\theta - \tan^2\theta}\right) \text{ but } \sec^2\theta - \tan^2\theta = 1$$

$$= \frac{2a\sec\theta}{2} = a\sec\theta$$

$$y_M = \frac{1}{2}\left(\frac{\cancel{b\sec\theta} + b\tan\theta \cancel{- b\sec\theta} + b\tan\theta}{\sec^2\theta - \tan^2\theta}\right) \text{ but } \sec^2\theta - \tan^2\theta = 1$$

$$= \frac{2b\tan\theta}{2} = b\tan\theta$$

So, $M_{QR} = (a\sec\theta, b\tan\theta) = P$ or $PQ=PR$

Alternate approaches use distance formula or geometric relationships using similar triangles or intercepts on parallel lines.

(c)	If α, β, γ are the roots of the equation $x^3 + 2x^2 + 3x - 4 = 0$, find a polynomial equation whose roots are:	
	(i) $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$	**2**

An equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$ is given by:

$$\left(\frac{1}{x}\right)^3 + 2\left(\frac{1}{x}\right)^2 + 3\left(\frac{1}{x}\right) - 4 = 0 \quad \text{multiply by } x^3$$

$$1 + 2x + 3x^2 - 4x^3 = 0 \quad \text{or} \quad 4x^3 - 3x^2 - 2x - 1 = 0$$

(ii) α^2, β^2 and γ^2	**2**

An equation with roots α^2, β^2 and γ^2 is given by:

$\left(\sqrt{x}\right)^3 + 2\left(\sqrt{x}\right)^2 + 3\left(\sqrt{x}\right) - 4 = 0$

$x\sqrt{x} + 2x + 3\sqrt{x} - 4 = 0$

$(x+3)\sqrt{x} = 4 - 2x$ square both sides

$\left(x^2 + 6x + 9\right)x = 16 + 4x^2 - 16x$

$x^3 + 2x^2 + 25x - 16 = 0$

(d)	A sequence is defined by $a_1 = 5,\ a_2 = 13$ and $a_{n+2} = 5a_{n+1} - 6a_n$ for all natural numbers n. Use Mathematical Induction to prove that $a_n = 2^n + 3^n$.	**3**

Let $S(n)$ be the statement $a_n = 2^n + 3^n$.

Test $n = 1$: $a_1 = 2^1 + 3^1 = 5$ So, $S(1)$ is true.

Test $n = 2$ $a_2 = 2^2 + 3^2 = 4 + 9 = 13$ So, $S(2)$ is true.

Assume $S(k)$ and $S(k+1)$ are true. i.e. assume $a_k = 2^k + 3^k$ and $a_{k+1} = 2^{k+1} + 3^{k+1}$

Prove $S(k+2)$ i.e prove $a_{k+2} = 2^{k+2} + 3^{k+2}$.

$\text{LHS} = a_{k+2}$

$= 5a_{k+1} - 6a_k$ using the given recursive formula

$= 5\left(2^{k+1} + 3^{k+1}\right) - 6\left(2^k + 3^k\right)$ by assumption

$= 5.2^{k+1} + 5.3^{k+1} - 6.2^k - 6.3^k$

$= 5.2^{k+1} - 3.2.2^k + 5.3^{k+1} - 2.3.3^k$

$= 5.2^{k+1} - 3.2^{k+1} + 5.3^{k+1} - 2.3^{k+1}$

$= 2.2^{k+1} + 3.3^{k+1}$

$= 2^{k+2} + 3^{k+2}$

$= \text{RHS}$

Hence, $S(k+2)$ is true.

So, $S(n)$ is true by Mathematical Induction.

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) Using the substitution $t = \tan\dfrac{x}{2}$, or otherwise, evaluate **3**

$$\int_0^{\frac{\pi}{3}} \frac{1}{1+\cos x - \sin x}\,dx \quad .$$

Let $t = \tan\dfrac{x}{2}$ then $x = 2\tan^{-1} t$ and $dx = \dfrac{2dt}{1+t^2}$

x	0	$\frac{\pi}{3}$
t	0	$\frac{1}{\sqrt{3}}$

Also, $\cos x = \dfrac{1-t^2}{1+t^2}$ and $\sin x = \dfrac{2t}{1+t^2}$

$$\begin{aligned}
\int_0^{\frac{\pi}{3}} \frac{1}{1+\cos x - \sin x}\,dx &= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+\dfrac{1-t^2}{1+t^2}-\dfrac{2t}{1+t^2}} \times \frac{2dt}{1+t^2} \\
&= \int_0^{\frac{1}{\sqrt{3}}} \frac{2dt}{1+t^2+1-t^2-2t} \\
&= \int_0^{\frac{1}{\sqrt{3}}} \frac{2dt}{2-2t} == \int_0^{\frac{1}{\sqrt{3}}} \frac{dt}{1-t} \\
&= \left[-\log(1-t)\right]_0^{1/\sqrt{3}} \\
&= -\log\left(1-\frac{1}{\sqrt{3}}\right) + \log 1 \\
&= -\log\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right) = \log\left(\frac{\sqrt{3}}{\sqrt{3}-1}\right)
\end{aligned}$$

(b) (i) Use integration by parts to find $\int x\tan^{-1} x\,dx$. **2**

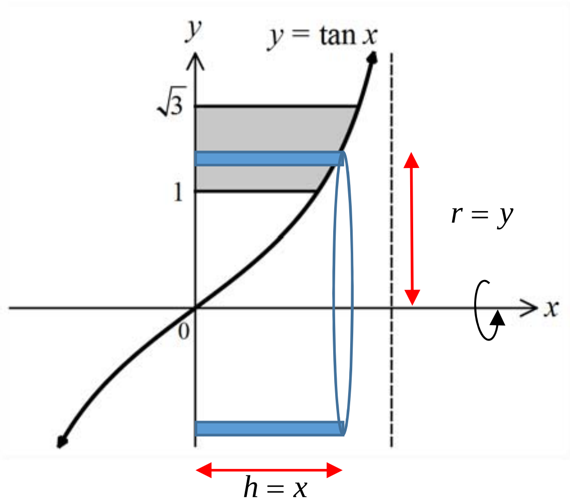
$$\int x\tan^{-1} x\,dx = \frac{x^2}{2}\tan^{-1} x - \int \frac{x^2}{2}.\frac{1}{1+x^2}\,dx \quad \text{using integration by parts}$$

$$\begin{aligned}
&= \frac{1}{2}\left[x^2\tan^{-1} x - \int \frac{x^2+1-1}{1+x^2}\,dx\right] \\
&= \frac{1}{2}\left[x^2\tan^{-1} x - \int \left(1-\frac{1}{1+x^2}\right)dx\right] \\
&= \frac{1}{2}\left[x^2\tan^{-1} x - x + \tan^{-1} x\right] + C \\
&= \frac{1}{2}\left[\left(x^2+1\right)\tan^{-1} x - x\right] + C
\end{aligned}$$

$u = \tan^{-1} x \qquad v' = x$

$u' = \dfrac{1}{1+x^2} \qquad v = \dfrac{x^2}{2}$

(ii) The region bounded by the curve $y = \tan x$, the lines $y = 1$, $y = \sqrt{3}$ and the y-axis is rotated about the x-axis. **2**



Use the method of cylindrical shells to find the volume of the solid of revolution formed.

Radius of shell is y and height of cylindrical shell is $x = \tan^{-1} y$. Thickness of shell is δy.

Volume of cylindrical shell is $\delta V = 2\pi y \tan^{-1} y \delta y$.

$$V = \lim_{\delta y \to 0} \sum_{y=1}^{\sqrt{3}} \delta V$$

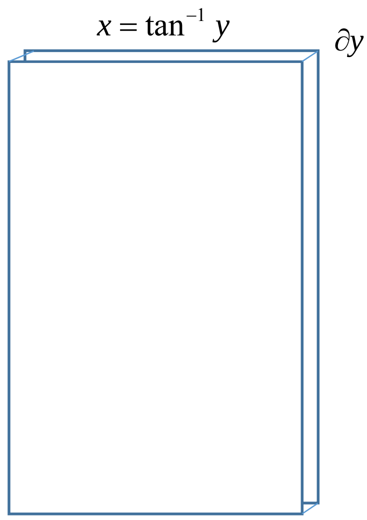
$$= 2\pi \int_1^{\sqrt{3}} y \tan^{-1} y\, dy$$

$$= 2\pi \frac{1}{2} \left[\left(y^2 + 1\right) \tan^{-1} y - y \right]_1^{\sqrt{3}}$$

$$= \pi \left[\left(4 \tan^{-1} \sqrt{3} - \sqrt{3}\right) - \left(2 \tan^{-1} 1 - 1\right) \right]$$

$$= \pi \left(\frac{4\pi}{3} - \sqrt{3} - 2\frac{\pi}{4} + 1 \right)$$

$$= \pi \left(\frac{5\pi}{6} - \sqrt{3} + 1 \right) \text{ units}^3$$



Cylindrical shell as rectangular prism

(c) (i) Show that $a^2 + 9b^2 \geq 6ab$, where a and b are real numbers. **1**

$a^2 + 9b^2 - 6ab = (a - 3b)^2 \geq 0$ for all real a, b (perfect square)

$\therefore a^2 + 9b^2 \geq 6ab$

(ii) Hence, or otherwise, show that $a^2 + 5b^2 + 9c^2 \geq 3(ab + bc + ac)$. **2**

$a^2 + 9b^2 \geq 6ab$ from (i)

Similarly, $b^2 + 9c^2 \geq 6bc$ and $a^2 + 9c^2 \geq 6ac$

Adding the three results we have:

$$2a^2+10b^2+18c^2 \geq 6(ab+bc+ac)$$
$$a^2+5b^2+9c^2 \geq 3(ab+bc+ac)$$

(iii)	Hence if $a>b>c>0$, show that $a^2+5b^2+9c^2>9bc$.	**1**

$a^2+5b^2+9c^2 \geq 3(ab+ac+bc)$ from (ii)

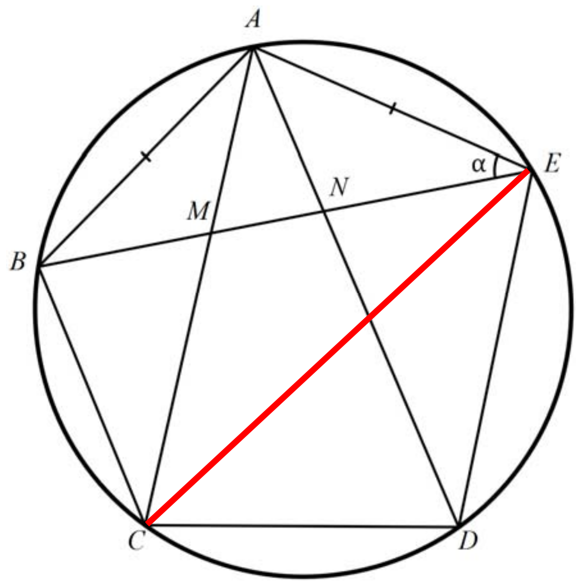
$> 3(bc+bc+bc)$ as $a>c \Rightarrow ab>bc$ and $a>b \Rightarrow ac>bc$

$=3(3bc)$

$=9bc$

$a^2+5b^2+9c^2>9bc$

(d) *ABCDE* is a pentagon inscribed in a circle and $AB=AE$.
BE meets *AC* and *AD* at *M* and *N* respectively. Let $\angle BEA=\alpha$.



(i)	Explain why $\angle ACE=\alpha$.	**1**

Join *CE*.
AB and *AE* are equal chords and hence subtend equal angles at the centre and hence at the circumference. Therefore, $\angle AEB=\angle ACE=\alpha$

(ii)	Prove that *CDNM* is a cyclic quadrilateral.	**3**

Let $\angle ECD=\beta$. Then $\angle ECD=\angle EAD=\beta$ (angles in same segment on chord *ED*)
$\angle MCD=\alpha+\beta$ (adding adjacent angles)
$\angle ANM=\alpha+\beta$ (exterior angle of ΔANE)
$\therefore \angle ANM=\angle MCD$
Then, *CDNM* is cyclic (exterior angle is equal to opposite interior angle)

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) Given that $p(x) = x^3 - 3ax^2 + b$ has a double zero, where a and b are non-zero real numbers, show that $4a^3 - b = 0$. **2**

Let $x = \alpha$ be the double zero. Then $p(\alpha) = p'(\alpha) = 0$ by the Multiple Root Theorem

$p'(x) = 3x^2 - 6ax$

$p'(\alpha) = 3\alpha^2 - 6a\alpha = 0$

$3\alpha(\alpha - 2a) = 0$

$\alpha = 2a$ Note $\alpha \neq 0$ as $P(0) = b \neq 0$

$p(\alpha) = \alpha^3 - 3a\alpha^2 + b = 0$

$(2a)^3 - 3a(2a)^2 + b = 0$

$8a^3 - 12a^3 + b = 0$

$-4a^3 + b = 0$ or $4a^3 - b = 0$

(b) $P\left(cp, \dfrac{c}{p}\right)$ lies on the hyperbola $xy = c^2$.

(i) Show that the equation of the normal to the hyperbola at P is given by **2**

$$px - \frac{y}{p} = c\left(p^2 - \frac{1}{p^2}\right).$$

$$x = cp \Rightarrow \frac{dx}{dp} = c$$

$$y = \frac{c}{p} \Rightarrow \frac{dy}{dp} = -\frac{c}{p^2}$$

$$\frac{dy}{dx} = \frac{dy/dp}{dx/dp} = \frac{-c/p^2}{c} = -\frac{1}{p^2}$$

Gradient of normal is p^2.

Equation of normal is:

$y - \dfrac{c}{p} = p^2(x - cp)$ divide by p

$$\frac{y}{p} - \frac{c}{p^2} = px - cp^2$$

$$px - \frac{y}{p} = cp^2 - \frac{c}{p^2}$$

$$px - \frac{y}{p} = c\left(p^2 - \frac{1}{p^2}\right)$$

	(ii)	The equation of the tangent at P is $x + p^2 y = 2cp$. DO NOT PROVE THIS. The normal at P cuts the x-axis at Q and the tangent at P cuts the y-axis at R. M is the midpoint of QR. Find the equation of the locus of M as P moves on the hyperbola.	**3**

$Q = \left(c\left(p - \frac{1}{p^3} \right), 0 \right)$ and $R = \left(0, \frac{2c}{p} \right)$

$M = \left(\frac{c}{2}\left(p - \frac{1}{p^3} \right), \frac{c}{p} \right)$

For locus of M,

$x = \frac{c}{2}\left(p - \frac{1}{p^3} \right)$ (1) and $y = \frac{c}{p}$ (2)

From (2), $p = \frac{c}{y}$. Sub into (1)

$x = \frac{c}{2}\left(\frac{c}{y} - \frac{y^3}{c^3} \right)$

$x = \frac{c^2}{2y} - \frac{y^3}{2c^2}$ multiply by $2c^2 y$

$2c^2 xy = c^4 - y^4$

(c)	Let $I = \displaystyle\int_{\frac{1}{a}}^{a} \frac{f(x)}{x\left(f(x) + f\left(\frac{1}{x} \right) \right)} dx.$		
	(i)	Use a suitable substitution to show that $I = \displaystyle\int_{\frac{1}{a}}^{a} \frac{f\left(\frac{1}{x} \right)}{x\left(f(x) + f\left(\frac{1}{x} \right) \right)} dx$.	**2**

Let $u = \frac{1}{x}$. Then $du = -\frac{1}{x^2} dx$

When $x = \frac{1}{a}$, $u = a$ and when $x = a$, $u = \frac{1}{a}$

$f(x) = f\left(\frac{1}{u} \right)$ and $f\left(\frac{1}{x} \right) = f(u)$

$$\int_{\frac{1}{a}}^{a} \frac{f(x)}{x\left(f(x)+f\left(\frac{1}{x}\right)\right)}dx. = \int_{a}^{\frac{1}{a}} \frac{f\left(\frac{1}{u}\right)}{\frac{1}{u}\left(f\left(\frac{1}{u}\right)+f(u)\right)}\left(-\frac{1}{u^2}du\right)$$

$$= \int_{\frac{1}{a}}^{a} \frac{f\left(\frac{1}{u}\right)}{u\left(f\left(\frac{1}{u}\right)+f(u)\right)}du$$

$$= \int_{\frac{1}{a}}^{a} \frac{f\left(\frac{1}{x}\right)}{x\left(f\left(\frac{1}{x}\right)+f(x)\right)}dx \qquad \text{as } u \text{ is a dummy variable}$$

(ii) Hence, or otherwise, evaluate $\displaystyle\int_{\frac{1}{2}}^{2} \frac{\sin x}{x\left(\sin x+\sin\frac{1}{x}\right)}dx$. **2**

$$\text{Let } I = \int_{\frac{1}{2}}^{2} \frac{\sin x}{x\left(\sin x+\sin\frac{1}{x}\right)}dx = \int_{\frac{1}{2}}^{2} \frac{\sin\frac{1}{x}}{x\left(\sin x+\sin\frac{1}{x}\right)}dx \qquad \text{using result in (i)}$$

$$2I = \int_{\frac{1}{2}}^{2} \frac{\sin x}{x\left(\sin x+\sin\frac{1}{x}\right)}dx + \int_{\frac{1}{2}}^{2} \frac{\sin\frac{1}{x}}{x\left(\sin x+\sin\frac{1}{x}\right)}dx$$

$$= \int_{\frac{1}{2}}^{2} \frac{\cancel{\sin x+\sin\frac{1}{x}}}{x\cancel{\left(\sin x+\sin\frac{1}{x}\right)}}dx$$

$$= \int_{\frac{1}{2}}^{2} \frac{1}{x}dx$$

$$= \left[\log_e x\right]_{1/2}^{2}$$

$$= \log 2 - \log\frac{1}{2} = \log 2 + \log 2$$

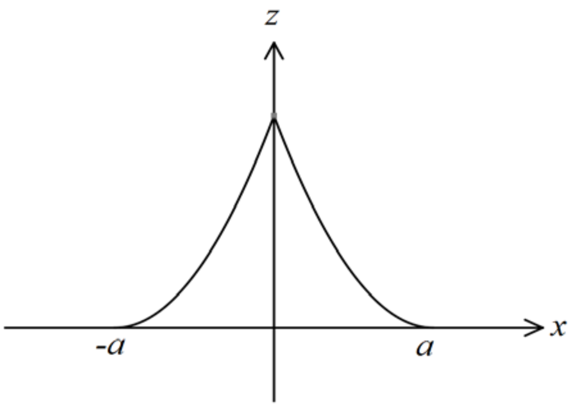
$$= 2\log 2$$

$$I = \log 2$$

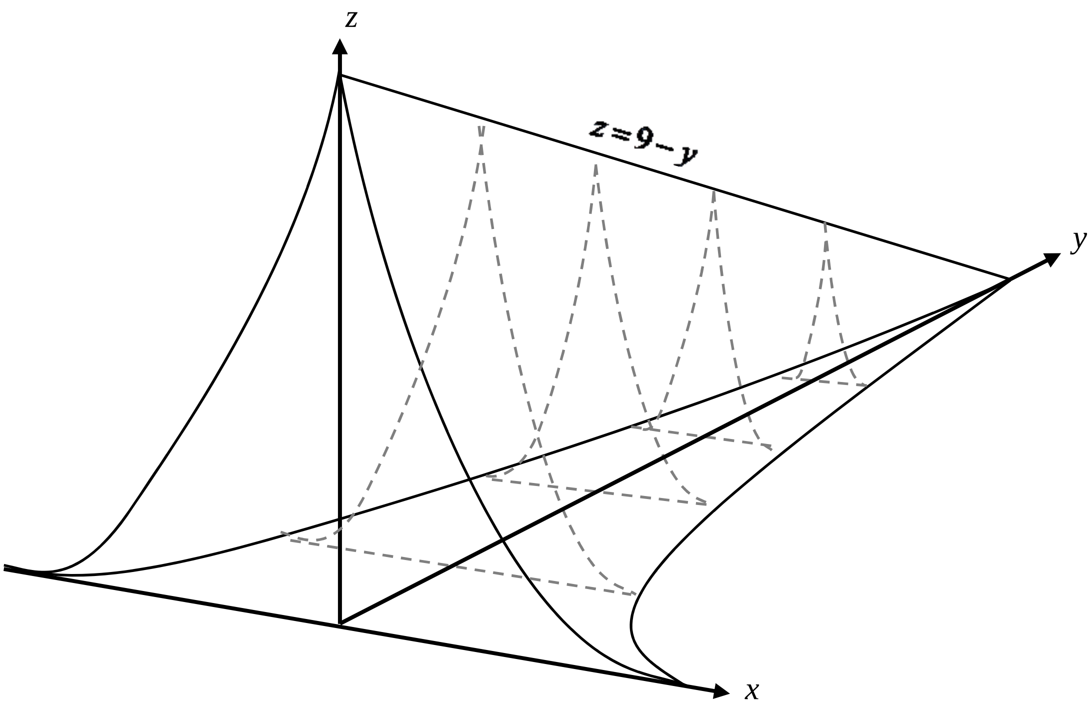
(d)	(i)	Find the area bounded by the curve $y=(x-a)^2$ and the coordinate axes, where $a>0$.	**1**

$$A=\int_0^a (x-a)^2\,dx$$
$$=\left[\frac{(x-a)^3}{3}\right]_0^a$$
$$=0-\left(\frac{-a^3}{3}\right)=\frac{a^3}{3}$$

	(ii)	Cross-sections of a solid perpendicular to the base are sections of two parabolas $z=(x-a)^2$ and $z=(x+a)^2$ as shown below.	**3**

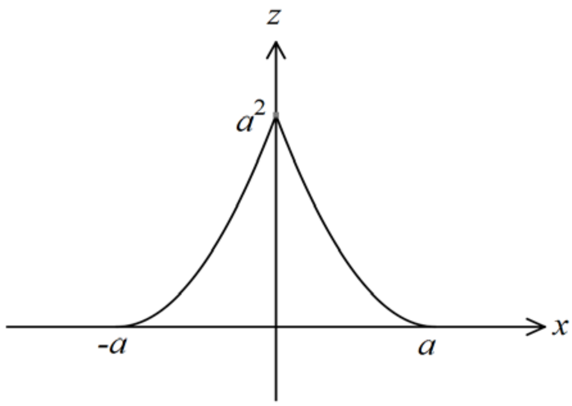


The heights of the cross-sections are bounded by the line $z=9-y$.



Find the volume of the solid.

Typical element:



Area of typical element is $\frac{2a^3}{3}$ from (i), where a is the x-intercept of the parabolic segment.

The height of the typical element is $9-y$ which equates to a^2 (the y-intercept of the parabolic segment). So $a=(9-y)^{\frac{1}{2}}$.

So, area of the typical element is $\frac{2}{3}(9-y)^{\frac{3}{2}}$.

Volume of typical element is $\delta V=\frac{2}{3}(9-y)^{\frac{3}{2}}\,\delta y$

$$\begin{aligned}
V &= \lim_{\delta y\to 0}\sum_{y=0}^{9}\delta V \\
&= \int_0^9 \frac{2}{3}(9-y)^{\frac{3}{2}}\,dy \\
&= \frac{2}{3}\left[-\frac{2}{5}(9-y)^{\frac{5}{2}}\right]_0^9 \\
&= \frac{-4}{15}\left(0-3^5\right) \\
&= \frac{324}{5}\text{ units}^3
\end{aligned}$$

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) Consider $I_n = \int (x^2+1)^{-n}\,dx, \quad n>0$.

(i) Show that $I_{n+1} = \dfrac{x(x^2+1)^{-n}}{2n} + \dfrac{2n-1}{2n} I_n$. **3**

Using integration by parts,

$$I_n = \int 1.(x^2+1)^{-n}\,dx$$

$$= x(x^2+1)^{-n} - \int -2nx^2(x^2+1)^{-(n+1)}\,dx$$

$$u = (x^2+1)^{-n} \qquad v' = 1$$

$$u' = -2nx(x^2+1)^{-(n+1)} \qquad v = x$$

$$I_n = x(x^2+1)^{-n} + 2n\int (x^2+1-1)(x^2+1)^{-(n+1)}\,dx$$

$$= x(x^2+1)^{-n} + 2n\int \left[(x^2+1)^{-n} - (x^2+1)^{-(n+1)}\right]dx$$

$$= x(x^2+1)^{-n} + 2nI_n - 2nI_{n+1}$$

Rearranging, we have

$$2nI_{n+1} = x(x^2+1)^{-n} + (2n-1)I_n$$

$$I_{n+1} = \frac{x(x^2+1)^{-n}}{2n} + \frac{(2n-1)}{2n}I_n$$

(ii) Find I_2. **1**

$$I_2 = \frac{x(x^2+1)^{-1}}{2} + \frac{1}{2}I_1 \qquad \text{using (i)}$$

$$= \frac{x}{2(x^2+1)} + \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$= \frac{x}{2(x^2+1)} + \frac{1}{2}\tan^{-1} x + C$$

(b) (i) Using the result $\cos(A+B)-\cos(A-B)=-2\sin A\sin B$, or otherwise, **2**

show that $\sin\theta\sum_{r=1}^{n}\sin 2r\theta=\sin(n+1)\theta\sin n\theta$.

$$\sin\theta\sum_{r=1}^{n}\sin 2r\theta=\sum_{r=1}^{n}\sin\theta\sin 2r\theta=-\frac{1}{2}\sum_{r=1}^{n}-2\sin\theta\sin 2r\theta$$

$$=-\frac{1}{2}\sum_{r=1}^{n}\left[\cos(2r+1)\theta-\cos(2r-1)\theta\right] \quad \text{using the given result}$$

$$=-\frac{1}{2}\left[\sum_{r=1}^{n}\cos(2r+1)\theta-\sum_{r=1}^{n}\cos(2r-1)\theta\right]$$

$$=-\frac{1}{2}\left[\sum_{r=1}^{n}\cos(2r+1)\theta-\sum_{r=0}^{n-1}\cos(2r+1)\theta\right]$$

$$=-\frac{1}{2}\left[\sum_{r=1}^{n-1}\cos(2r+1)\theta+\cos(2n+1)\theta-\cos\theta-\sum_{r=1}^{n-1}\cos(2r+1)\theta\right]$$

$$=-\frac{1}{2}\left[\cos(2n+1)\theta-\cos\theta\right]$$

$$=-\frac{1}{2}\left[\cos(n\theta+\theta+n\theta)-\cos(n\theta+\theta-n\theta)\right]$$

$$=-\frac{1}{2}\left[-2\sin(n\theta+\theta)\sin n\theta\right] \quad \text{using the given result}$$

$$=\sin(n+1)\theta\sin n\theta \quad \text{as required}$$

(ii) Hence show that $\sum_{r=1}^{8}\sin\frac{r\pi}{9}=\cot\frac{\pi}{18}$. **2**

Using $\theta=\frac{\pi}{18}$ and $n=8$ in (i) we have

$$\sin\frac{\pi}{18}\sum_{r=1}^{8}\sin.\frac{2r\pi}{18}=\sin 9\left(\frac{\pi}{18}\right)\sin 8\left(\frac{\pi}{18}\right)$$

$$\sin\frac{\pi}{18}\sum_{r=1}^{8}\sin.\frac{r\pi}{9}=\sin\frac{\pi}{2}\sin\left(\frac{4\pi}{9}\right)$$

$$\sum_{r=1}^{8}\sin.\frac{r\pi}{9}=\frac{\sin\frac{\pi}{2}\sin\left(\frac{4\pi}{9}\right)}{\sin\frac{\pi}{18}}=\frac{\sin\left(\frac{4\pi}{9}\right)}{\sin\frac{\pi}{18}}$$

$$=\frac{\cos\left(\frac{\pi}{2}-\frac{4\pi}{9}\right)}{\sin\frac{\pi}{18}}=\frac{\cos\frac{\pi}{18}}{\sin\frac{\pi}{18}}$$

$$=\cot\frac{\pi}{18}$$

(c)	(i)	Show that $\log_e x \le x - 1$ for $x > 0$.	**2**

Let $f(x) = \log x - x + 1$.

$$f'(x) = \frac{1}{x} - 1 = \frac{1-x}{x}$$

Stationary point at $f'(x) = 0 \Rightarrow x = 1$

$f(1) = 0$. So, stationary point at $(1,0)$.

$$f''(x) = -\frac{1}{x^2} < 0 \text{ for all } x .$$

So the function is always concave down and $(1,0)$ is a maximum turning point.

$\therefore f(x) = \log x - x + 1 \le 0$ for all x in its domain ie for $x > 0$.

So $\log x - x + 1 \le 0$ or $\log x \le x - 1$

	(ii)	Show that $\log_e\left(\frac{c_1c_2...c_n}{\mu^n}\right) \le \frac{c_1 + c_2 + ... + c_n}{\mu} - n$, where $c_1, c_2, ..., c_n > 0$ and $\mu > 0$.	**2**

$$\begin{aligned}
\log\left(\frac{c_1c_2...c_n}{\mu^n}\right) &= \log\left(\frac{c_1}{\mu}.\frac{c_2}{\mu}...\frac{c_n}{\mu}\right) \\
&= \log\left(\frac{c_1}{\mu}\right) + \log\left(\frac{c_2}{\mu}\right) + ... + \log\left(\frac{c_n}{\mu}\right) \\
&\le \left(\frac{c_1}{\mu} - 1\right) + \left(\frac{c_2}{\mu} - 1\right) + ... + \left(\frac{c_n}{\mu} - 1\right) \quad \text{using (i)} \\
&= \frac{c_1}{\mu} + \frac{c_2}{\mu} + ... + \frac{c_n}{\mu} - n \\
&= \frac{c_1 + c_2 + ... + c_n}{\mu} - n
\end{aligned}$$

	(iii)	Hence if $\mu = \frac{c_1 + c_2 + ... + c_n}{n}$, show that $\sqrt[n]{c_1c_2...c_n} \le \frac{c_1 + c_2 + ... + c_n}{n}$.	**2**

Using $\mu = \frac{c_1 + c_2 + ... + c_n}{n}$ in the result from (ii)

$$\log\left(\frac{c_1c_2...c_n}{\left(\frac{c_1 + c_2 + ... + c_n}{n}\right)^n}\right) \le \frac{c_1 + c_2 + ... + c_n}{\left(\frac{c_1 + c_2 + ... + c_n}{n}\right)} - n$$

$$\log\left(\frac{n^n c_1c_2...c_n}{(c_1 + c_2 + ... + c_n)^n}\right) \le n - n = 0$$

$$\frac{n^n c_1 c_2 \ldots c_n}{\left(c_1 + c_2 + \ldots + c_n\right)^n} \le 1$$

$$n^n c_1 c_2 \ldots c_n \le \left(c_1 + c_2 + \ldots + c_n\right)^n$$

$$c_1 c_2 \ldots c_n \le \left(\frac{c_1 + c_2 + \ldots + c_n}{n}\right)^n$$

$$\sqrt[n]{c_1 c_2 \ldots c_n} \le \frac{c_1 + c_2 + \ldots + c_n}{n}$$

$$\frac{c_1 + c_2 + \ldots + c_n}{n} \ge \sqrt[n]{c_1 c_2 \ldots c_n}$$

(iv)	Hence use part (iii) to find a lower bound for $\frac{101}{103} + \frac{103}{105} + \frac{105}{107} + \ldots. + \frac{197}{199} + \frac{199}{101}$.	**1**

Using result (iii) with $c_1 = \frac{101}{103}; c_2 = \frac{103}{105}; c_3 = \frac{105}{107} \ldots c_{50} = \frac{199}{101}$

$$\frac{\frac{101}{103} + \frac{103}{105} + \ldots + \frac{197}{199} + \frac{199}{101}}{50} \ge \sqrt[50]{\frac{101}{103} \times \frac{103}{105} \times \ldots \times \frac{197}{199} \times \frac{199}{101}} = 1$$

$$\frac{101}{103} + \frac{103}{105} + \ldots + \frac{197}{199} + \frac{199}{101} \ge 50$$

End of solutions